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Symmetries of Spherical Harmonics: applications to ambisonics

Michael Chapman[1]

[1] 01350 Culoz, France (chapman@mchapman.com)

Abstract: *Spherical harmonics are often represented as images of three dimensional forms. Inspection shows that mirroring these (here, a reflection in the planes of the three axis pairs) frequently results in no change. That is symmetries exist.*
Using direction cosine formulæ for the spherical harmonics these symmetries are established and then extrapolated to higher degrees.
The results are used

- *to produce formulae generalised for an any order signal set for reflections about the planes,*
- *to produce 'skeleton' matrices for pitch and roll, rotations in 90^o steps, for any ambisonic order, and,*
- *to prove algebraicly that dominance –for practical purposes– is uniquely a first–order transformation*

It is suggested that this approach may also be useful in an analysis of 'redundant channels' in decoding.
An extension from consideration of transformations in steps of 90^o to 'any angle' transformations provides a surprisingly simple (though currently tedious) method for deriving transformation matrices for real spherical harmonics.

Key words: symmetry, direction cosine, spherical harmonic transformations.

1 INTRODUCTION

It is only slightly overstating the case to say that physics is the study of symmetry P.W. Anderson[2]

Spherical harmonics are usually described in terms of their degree[3] (l) and order[4] (m) within that degree, written as Y_l^m.

The channels in an ambisonic signal set are each related to a specific spherical harmonic, and can be written[5] as B_l^m. For many uses, though, assigning a unique integer to each channel is more convenient and the channel can then be described[6] as B_n. This, the *ambisonic channel number* (ACN), is related to (l, m) as $n = l(l+1) + m$. Spherical harmonics may likewise be notated as Y_n. (See Chapman & Cotterell's paper to this Symposium [4] for a more detailed description.)

In figure 1 it will be seen that reflection through the x–y plane, the y–z plane or the z–x plane frequently results in no change to the spherical harmonic.

This trivial observation is related to various common facts about transformations of ambisonic signal sets. For example B_0 is invariant under any rotation. To generalise from these observations we turn to the equations for the spherical harmonics illustrated in figure 1.

[2] Nobel laureate. The quotation is attributed to his 1972 essay "More is Different".

[3] l, any integer ≥ 0.

[4] Not to be confused with *ambisonic order*!
m an integer such that $-l \leq m \leq l$.

[5] In this paper we use B to represent a component of the signal set $\mathbf{B}$. That is bold upper case letters represent matrices. We will also use B', $\mathbf{B}'$, etc. to represent a component and a signal set that have been transformed.

[6] n is an integer and ≥ 0.

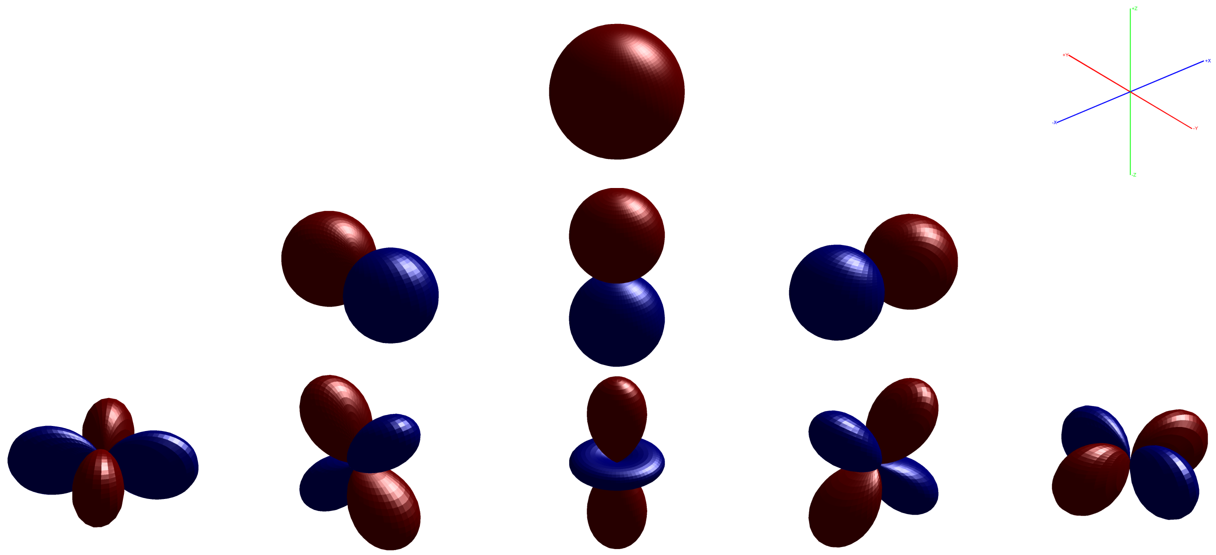

Figure 1: Graphic representation of the early spherical harmonics. The axes are x (blue), y (red) and z (green) with positive directions up/away from the reader. Rows are $l = 0$, $l = 1$ and $l = 2$. Columns are $m = -2$ to $m = +2$. Thus the SHs illustrated are Y_0 / Y_1, Y_2, Y_3 / Y_4, Y_5, Y_6, Y_7 and Y_8. Graphic courtesy of and © Bruce Wiggins, Signal Processing Applications Group, University of Derby.

2 DIRECTION COSINES

The equations for spherical harmonics are generally expressed as functions of azimuth (θ) and elevation (ϕ), for example (in N3D [8]):

$$Y_4 = \sqrt{15} \cdot \cos\theta \cdot \sin\theta \cdot \cos^2\phi$$

now this[7] may equally well be written in terms of direction cosines:

$$Y_4 = \sqrt{15} \cdot u_x \cdot u_y$$

where [10]

$$\vec{u} = \left| \begin{array}{lcl} u_x & = & \cos(\theta) \cdot \cos(\phi) \\ u_y & = & \sin(\theta) \cdot \cos(\phi) \\ u_z & = & \sin(\phi) \end{array} \right. \tag{1}$$

Generically one might write:

$$Y_n = a_n(\theta, \phi)$$

where a_n is a function, or for direction cosines:

$$Y_n = b_n(u_x, u_y, u_z) \tag{2}$$

where b_n is a function.

Early values, for direction cosine forms, are widely published (e.g. Daniel [11], upto third order), Fons Adriaensen [7] kindly supplied the author with values for fourth

Y_0	$=$	1
Y_1	$=$	$\sqrt{3} \cdot u_y$
Y_2	$=$	$\sqrt{3} \cdot u_z$
Y_3	$=$	$\sqrt{3} \cdot u_x$
Y_4	$=$	$\sqrt{15} \cdot u_x \cdot u_y$
Y_5	$=$	$\sqrt{15} \cdot u_y \cdot u_z$
Y_6	$=$	$\frac{\sqrt{5}}{2} \cdot (2u_z^2 - u_x^2 - u_y^2)$
Y_7	$=$	$\sqrt{15} \cdot u_x \cdot u_z$
Y_8	$=$	$\frac{\sqrt{15}}{2} \cdot (u_x^2 - u_y^2)$
Y_9	$=$	$\sqrt{\frac{35}{8}} \cdot u_y \cdot (3u_x^2 - u_y^2)$
Y_{10}	$=$	$\sqrt{105} \cdot u_z \cdot u_x \cdot u_y$
Y_{11}	$=$	$\sqrt{\frac{21}{8}} \cdot u_y \cdot (4u_z^2 - u_x^2 - u_y^2)$
Y_{12}	$=$	$\frac{\sqrt{7}}{2} \cdot u_z \cdot (2u_z^2 - 3u_x^2 - 3u_y^2)$
Y_{13}	$=$	$\sqrt{\frac{21}{8}} \cdot u_x \cdot (4u_z^2 - u_x^2 - u_y^2)$
Y_{14}	$=$	$\frac{\sqrt{105}}{2} \cdot u_z \cdot (u_x^2 - u_y^2)$
Y_{15}	$=$	$\sqrt{\frac{35}{8}} \cdot u_x \cdot (u_x^2 - 3u_y^2)$

Table 1: Direction cosine forms of the spherical harmonic equations used in ambisonics, from the work of Philip Cotterell [5]. In N3D [8], and here converted so terms have equal powers (see text for explanation). (The tabulation continues in table 2.)

[7] This is often further ‘simplified’ as $\cos\theta \cdot \sin\theta = (1/2)\sin(2\theta)$.

degree, and Philip Cotterell's work [2] has extended the published figures to sixth order. Thomas Musil [6] has independently published values upto fifth order.

There is a general pattern to the the direction cosine formulæ (tables 1 and 2), it may be expressed as

$$Y_n = k_n \cdot f_n(u_x, u_y, u_z) \cdot g_n(u_x^2, u_y^2, u_z^2)$$

where:

- k_n is purely numerical.
  (It is of no interest here; but possibly worth remarking, that k_n is always of the form $(a/b) \cdot \sqrt{c/d}$ where a, b, c, d are positive, non-zero, integers.)
- f_n is of the form $u_x^p \cdot u_y^q \cdot u_z^r$, where p, q and r may each be either 0 or 1. This part is discussed in further detail below.
- g_n is a polynomial of u_x^2, u_y^2 and u_z^2. As such, then inverting (multiplying by -1) any combination of u_x, u_y and u_z will not change the value of g_n.
  (This part may be written in a multitude of ways … —as $u_x^2 + u_y^2 + u_z^2 = 1$, always. See below for a fuller discussion.)

The prime attraction of direction cosines is that k_n can be pre-calculated and then determining the variable part requires no square roots and the only trigonometric calls are to obtain values for the three variables, all the rest of the calculation is simple multiplication, addition and subtraction and thus relatively speedy.

Our interest here, though, is the ability to split the variable part in two, and in particular to then study f_n.

$\mathbf{f_n}$

f_n must be one of the following:

$$\begin{gathered} 1 \\ u_z \\ u_y \\ u_y \cdot u_z \\ u_x \\ u_x \cdot u_z \\ u_x \cdot u_y \\ u_x \cdot u_y \cdot u_z \end{gathered}$$

For convenience of notation we will frequently substitute x for u_x, etc., in this paper.[8] It will be seen that 1 describes a spherical harmonic (SH) that is invariant whichever of the three axis planes it is reflected through. Likewise x describes a SH that is invariant when reflected through the *x–y plane* or *z–x* plane —put another way it is invariant *except* for reflection through the *y–z plane*, that is unless the *x-axis* is inverted. So the letters in our xyz notation describe the axes which if inverted do transform the SH.

[8] This has some justification beyond convenience. Green [16], for example, uses x/r, y/r and z/r and comments (page 22) "usually $r = 1$". Readability favours the Cartesian coordinates of a point on what is a unit sphere, rather than direction cosines of a unit vector. Practically they are the same.

The utility of this approach can be seen in deriving mirrored soundfields. If we wish to *flip* a soundfield (that is swap left and right, with up, down, front, back unchanged) then inverting (multiplying by -1) all the channels whose SH equations have u_y in f_n (that is the third, fourth, seventh and eighth in the list above) will achieve the desired mirroring.

(Likewise a *flop* (swapped front and back) is achieved by negating those with u_x, and a *flap* (swapped up and down) is achieved by negating those with u_z.)

It should be noted that Sontacchi, Zotter & Höldrich [14] used a similar approach when considering domed loudspeaker arrays (see figure 2 later in this paper).

$\mathbf{g_n}$

Our initial interest is f_n, but later in extending the concepts here we will return to g_n, and so some remarks are made about its form.

As $u_x^2 + u_y^2 + u_z^2 = 1$ then a term of any (even) degree can be obtained by simply multiplying by $u_x^2 + u_y^2 + u_z^2$ —that is by multiplying by 1.

There is though a (or possibly, at least one) simplest rendering where:

$$g_n = \sum_{i=1}^{j} k_i \cdot u_x^{2s_i} \cdot u_y^{2t_i} \cdot u_z^{2v_i} \tag{3}$$

where

$$p + q + r + 2(s_i + t_i + v_i) = l \tag{4}$$

that is the equation can be de–simplified to a series of terms, each having a power equal to the degree of the SH. This approach has a certain 'simplicity' to it, it is introduced as it also aids calculations.

As u_x^2, u_y^2 and u_z^2 are not independent of each other, then –as previously commented– each SH can be written in a multitude of ways. The most obvious example is perhaps for Y_l^0. These are all functions solely of $\sin(\phi)$ in (θ, ϕ) notation, and as $u_z = \sin(\phi)$ they can rapidly be derived from the (θ, ϕ) versions. This is not only convenient, but also demonstrates the relationship between the two formats. To take the simplest example (Y_6):

Y_{16}	$=$	$\frac{3}{2} \cdot \sqrt{35} \cdot u_x \cdot u_y \cdot (u_x^2 - u_y^2)$
Y_{17}	$=$	$\frac{3}{2} \cdot \sqrt{\frac{35}{2}} \cdot u_y \cdot u_z \cdot (3u_x^2 - u_y^2)$
Y_{18}	$=$	$\frac{3}{2} \cdot \sqrt{5} \cdot u_x \cdot u_y \cdot (6u_z^2 - u_x^2 - u_y^2)$
Y_{19}	$=$	$\frac{3}{2} \cdot \sqrt{\frac{5}{2}} \cdot u_y \cdot u_z \cdot (4u_z^2 - 3u_x^2 - 3u_y^2)$
Y_{20}	$=$	$\frac{3}{8} \cdot (8u_z^4 + 3u_x^4 + 3u_y^4 - 24u_x^2u_z^2 - 24u_y^2u_z^2 + 6u_x^2u_y^2)$
Y_{21}	$=$	$\frac{3}{2} \cdot \sqrt{\frac{5}{2}} \cdot u_x \cdot u_z \cdot (4u_z^2 - 3u_x^2 - 3u_y^2)$
Y_{22}	$=$	$\frac{3}{4} \cdot \sqrt{5} \cdot (u_x^2 - u_y^2) \cdot (6u_z^2 - u_x^2 - u_y^2)$
Y_{23}	$=$	$\frac{3}{2} \cdot \sqrt{\frac{35}{2}} \cdot (u_x^2 - 3u_y^2) \cdot u_x \cdot u_z$
Y_{24}	$=$	$\frac{3}{8} \cdot \sqrt{35} \cdot (u_x^4 - 6u_x^2 \cdot u_y^2 + u_y^4)$
Y_{25}	$=$	$\frac{3}{8} \cdot \sqrt{\frac{77}{2}} \cdot u_y \cdot (5u_x^4 - 10u_x^2u_y^2 + u_y^4)$
Y_{26}	$=$	$\frac{3}{2} \cdot \sqrt{385} \cdot u_x \cdot u_y \cdot u_z \cdot (u_x^2 - u_y^2)$
Y_{27}	$=$	$\frac{1}{8} \cdot \sqrt{\frac{385}{2}} \cdot u_y \cdot (u_y^4 - 2u_x^2u_y^2 - 3u_x^4 - 8u_y^2u_z^2 + 24u_x^2u_z^2)$
Y_{28}	$=$	$\frac{\sqrt{1155}}{2} \cdot u_x \cdot u_y \cdot u_z \cdot (2u_z^2 - u_x^2 - u_y^2)$
Y_{29}	$=$	$\frac{\sqrt{165}}{8} \cdot u_y \cdot (u_x^4 + 2u_x^2u_y^2 + u_y^4 - 12u_x^2u_z^2 - 12u_y^2u_z^2 + 8u_z^4)$
Y_{30}	$=$	$\frac{\sqrt{11}}{8} \cdot u_z \cdot (15u_x^4 + 15u_y^4 + 8u_z^4 + 30u_x^2u_y^2 - 40u_x^2u_z^2 - 40u_y^2u_z^2)$
Y_{31}	$=$	$\frac{\sqrt{165}}{8} \cdot u_x \cdot (u_x^4 + 2u_x^2u_y^2 + u_y^4 - 12u_x^2u_z^2 - 12u_y^2u_z^2 + 8u_z^4)$
Y_{32}	$=$	$\frac{\sqrt{1155}}{4} \cdot u_z \cdot (2u_x^2u_z^2 - 2u_y^2u_z^2 - u_x^4 + u_y^4)$
Y_{33}	$=$	$\frac{1}{8} \cdot \sqrt{\frac{385}{2}} \cdot u_x \cdot (2u_x^2u_y^2 + 8u_x^2u_z^2 - 24u_y^2u_z^2 - u_x^4 + 3u_y^4)$
Y_{34}	$=$	$\frac{3}{8} \cdot \sqrt{385} \cdot u_z \cdot (u_x^4 - 6u_x^2u_y^2 + u_y^4)$
Y_{35}	$=$	$\frac{3}{8} \cdot \sqrt{\frac{77}{2}} \cdot u_x \cdot (u_x^4 - 10u_x^2u_y^2 + 5u_y^4)$
Y_{36}	$=$	$\frac{1}{8} \cdot \sqrt{\frac{3003}{2}} \cdot u_x \cdot u_y \cdot (3u_x^4 - 10u_x^2u_y^2 + 3u_y^4)$
Y_{37}	$=$	$\frac{3}{8} \cdot \sqrt{\frac{1001}{2}} \cdot u_y \cdot u_z \cdot (5u_x^4 - 10u_x^2u_y^2 + u_y^4)$
Y_{38}	$=$	$\frac{3}{4} \cdot \sqrt{91} \cdot u_x \cdot u_y \cdot (-u_x^4 + u_y^4 + 10u_x^2u_z^2 - 10u_y^2u_z^2)$
Y_{39}	$=$	$\frac{1}{8} \cdot \sqrt{\frac{1365}{2}} \cdot u_y \cdot u_z \cdot (-9u_x^4 + 3u_y^4 - 6u_x^2u_y^2 + 24u_x^2u_z^2 - 8u_y^2u_z^2)$
Y_{40}	$=$	$\frac{1}{8} \cdot \sqrt{\frac{1365}{2}} \cdot u_x \cdot u_y \cdot (u_x^4 + u_y^4 + 16u_z^4 + 2u_x^2u_y^2 - 16u_x^2u_z^2 - 16u_y^2u_z^2)$
Y_{41}	$=$	$\frac{\sqrt{273}}{8} \cdot u_y \cdot u_z \cdot (5u_x^4 + 5u_y^4 + 8u_z^4 + 10u_x^2u_y^2 - 20u_x^2u_z^2 - 20u_y^2u_z^2)$
Y_{42}	$=$	$\frac{\sqrt{13}}{16} \cdot (16u_z^6 - 5u_x^6 - 5u_y^6 - 15u_x^4u_y^2 + 90u_x^4u_z^2 + 90u_y^4u_z^2 - 120u_y^2u_z^4 - 15u_x^2u_y^4 - 120u_x^2u_z^4 + 180u_x^2u_y^2u_y^2)$
Y_{43}	$=$	$\frac{\sqrt{273}}{8} \cdot u_x \cdot u_z \cdot (5u_x^4 + 5u_y^4 + 8u_z^4 + 10u_x^2u_y^2 - 20u_x^2u_z^2 - 20u_y^2u_z^2)$
Y_{44}	$=$	$\frac{1}{16} \cdot \sqrt{\frac{1365}{2}} \cdot (u_x^6 - u_y^6 + u_x^4u_y^2 - u_x^2u_y^4 - 16u_x^4u_z^2 + 16u_x^2u_z^4 + 16u_y^4u_z^2 - 16u_y^2u_z^4)$
Y_{45}	$=$	$\frac{1}{8} \cdot \sqrt{\frac{1365}{2}} \cdot u_x \cdot u_z \cdot (-3u_x^4 + 6u_x^2u_y^2 + 8u_x^2u_z^2 - 24u_y^2u_z^2 + 9u_y^4)$
Y_{46}	$=$	$\frac{3}{16} \cdot \sqrt{91} \cdot (-u_x^6 + 5u_x^4u_y^2 + 10u_x^4u_z^2 + 5u_x^2u_y^4 + 10u_y^4u_z^2 - u_y^6 - 60u_x^2u_y^2u_z^2)$
Y_{47}	$=$	$\frac{3}{8} \cdot \sqrt{\frac{1001}{2}} \cdot u_x \cdot u_z \cdot (u_x^4 - 10u_x^2u_y^2 + 5u_y^4)$
Y_{48}	$=$	$\frac{1}{16} \cdot \sqrt{\frac{3003}{2}} \cdot (u_x^6 - 15u_x^4u_y^2 + 15u_x^2u_y^4 - u_y^6)$

Table 2: Continuation of table 1. (To the author’s knowledge no equations for higher values are published.)

$$
\begin{aligned}
Y_6 &= \frac{\sqrt{5}}{2}\cdot(3\sin^2\phi - 1) &(5)\\
&= \frac{\sqrt{5}}{2}\cdot(3u_z^2 - 1) &(6)\\
&= \frac{\sqrt{5}}{2}\cdot(2u_z^2 - u_x^2 - u_y^2) &(7)
\end{aligned}
$$

(Where equation 6 shows its origins from 5, whilst 7 is what we here term 'equal powers' which is convenient for manipulations used later.)

In performing calculations using g_n one is forced to consider whether u_x, u_y and u_z are independent. Three variables are not required, as any position can be specified by two variable (e.g. (θ, ϕ)), but no two of (u_x, u_y, u_z) is sufficient. Initially, in the calculations presented below, one of the three squares was removed, e.g. u_z^2 (as $u_z^2 = 1 - u_x^2 - u_y^2$). In practice no simplification results, as zero-power terms (simple integers) appear on both sides of the equation and the solution is the same.

3 GENERALISING

At the time of writing direction cosine SH equations are only published upto sixth degree (tables 1 and 2). They are (currently very) tedious to work out. However if we ignore g_n (and k_n) then f_n does follow a pattern.

Upto sixth degree (and using x for u_x, etc.), this is shown in table 3.

The pattern is:

		l is odd	l is even
$m < 0$	and odd	u_y	$u_y.u_z$
	and even	$u_x.u_y.u_z$	$u_x.u_y$
$m \geq 0$	and odd	u_x	$u_x.u_z$
	and even	u_z	1

or[9]

	l	m
1	even	≥ 0 and even
u_x		< 0 and odd
		≥ 0 and even
u_y		< 0
u_z	odd	even
	even	odd

The parity of m and of the ACN are always the same.[10]

[9] In programming this is easily dealt with by `if, elsif, else` statements (see the `shake` subroutine in Ambisuite [13], for example.)

[10] As l is an integer, then either l or $l+1$ must be even, therefore $l(l+1)$ must be even, and $l(l+1)+m$ (the ACN) must be even if m is even, etc.

Example

Here we give a trivial example for second–degree.

First we recapitulate and set out the values of f_n for second–degree.

Y_n	f_n
Y_4	xy
Y_5	yz
Y_6	1
Y_7	xz
Y_8	1

example — 90^o

For a 90^o roll then z in output signals, that is z', must be replaced by y in input signals, that is y. Likewise y' is replaced by $-z$, and we might write:

$$
\begin{aligned}
y' &\leftarrow -z\\
z' &\leftarrow y\\
x' &\leftarrow x
\end{aligned}
$$

- So B_4' (the transformed B_4) will have $x'y'$, that is $-xz$ the only input signal it can acquire xz from is B_7, and it must be inverted.
- B_5' will have $y'z'$, that is $-yz$ the only input signal it can acquire yz from is B_5, and it must be inverted.
- B_7' will have $x'z'$, that is xy the only input signal it can acquire xy from is B_4.
- B_6' and B_8' remain respectively B_6 and B_8. (It can be proved that they do not take part of each other's signal, but suffice it here to say that we are only changing signals with y or z in the f_n of their underlying SH.)

So we can write:

$$
\begin{pmatrix}
0 & 0 & 0 & -1 & 0\\
0 & -1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0\\
1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \tag{8}
$$

which is the classic 90^o roll matrix for second degree.

Condon-Shortley phase

The Condon-Shortley phase occurs in some definitions of the spherical harmonics, but *not* in the definition used in ambisonics. Unfortunately some convenient, but in the final analysis inappropriate, software use this convention and have littered ambisonic literature with erroneous results. The most frequent, perhaps, being transformations that 'pitch' soundfields in the wrong direction. As Weisstein [15] comments: It "is not necessary in the definition

l \ m	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
0							1						
1						y	z	x					
2					xy	yz	1	xz	1				
3				y	xyz	y	z	x	z	x			
4			xy	yz	xy	yz	1	xz	1	xz	1		
5		y	xyz	y	xyz	y	z	x	z	x	z	x	
6	xy	yz	xy	yz	xy	yz	1	xz	1	xz	1	xz	1

Table 3: The values of f_n for the first seven degrees of spherical harmonics.

of the spherical harmonics, but including it simplifies the treatment of angular moment in quantum mechanics."

It is a phase factor of -1^m, which for our purposes is equivalent to -1^n —where n is the ACN (see footnote 10). That is every other (every odd) SH is negated. The effect on ambisonic signal sets is the same as inverting x and y. That is applying (or removing) this unwanted factor can be represented as:

$$\begin{array}{ccc} y' & \leftarrow & -y \\ z' & \leftarrow & z \\ x' & \leftarrow & -x \end{array}$$

This digression should not have been necessary. The problems due to inappropriate tools are though too common and a note on both how to recognise them and how to correct them may be of use to some readers.

4 USES

mirroring

So to *flip* (left-right) negate (inverse) all the channels with $m < 0$. To *flop* (front-back) inverse all channels that have (l, m) that match m is (< 0 and even) or (≥ 0 and odd). To *flop* (up-down) inverse all channels that have (l, m) such that (odd, even) or (even, odd), that is $l + m$ is odd.

A *flip* followed by a *flop* (or vice versa —transformations are generally not commutative, this is an exception) is equivalent to a 180^o rotation (a *yaw* rotation). So channels with f_n matching on u_x or u_y are inverted (the third, fourth fifth and sixth in the list above). Those matching on both u_x and u_y are inverted twice ... that is remain unchanged (seventh and eighth). The results of this operation can be compared with standard published yaw rotation matrices.

rotation

Determining pitch and roll matrices for 90^o (or 270^o) has proved difficult. A solution is due to be published (by Franz Zotter). His method will allow an arbitrary rotation to any new position, using only yaws ($\mathbf{R_z}$), which are easy to calculate for any degree, based on the relationship:

$$\mathbf{R}(\alpha, \beta, \gamma) = \mathbf{R_z}(\alpha) \times \mathbf{R_y}(\beta) \times \mathbf{R_z}(\gamma) \tag{9}$$

$$= \mathbf{R_z}(\alpha{+}45) \times \mathbf{R_y}(90) \times \mathbf{R_z}(\beta{+}180) \times \mathbf{R_y}(90) \times \mathbf{R_z}(\gamma{+}45) \tag{10}$$

and his precalculated tabulations of pitches ($\mathbf{R_y}$) for set angles. The latter are published [20], with a more detailed explanation [21] of the above.

The direction cosine method allows the generation of skeleton matrices, and is offered here.

For third degree, the signal set $\mathbf{B}$ consists of channels (ACNs) 9 to 15, these, respectively, have 'first part's that contain y, xyz, y, z, x, z and x, respectively.

For a 270^o pitch then:

$$\begin{array}{ccc} y' & \leftarrow & y \\ z' & \leftarrow & x \\ x' & \leftarrow & -z \end{array}$$

implying our matrix must be of the form:

$$\begin{pmatrix} * & 0 & * & 0 & 0 & 0 & 0 \\ 0 & -* & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -* & 0 & * \\ 0 & 0 & 0 & * & 0 & -* & 0 \\ 0 & 0 & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & -* & 0 & -* & 0 \end{pmatrix}$$

where $*$ represents a positive number (and indeed where the sum of the squares of the values in each column and each row is one ... thus the value for column 2, row 2 must be -1).

Extending this to find the missing values, whilst interesting, is futile[11] as Zotter's method can be automated, whilst this method does not seem prone to automation.

[11] One can tediously work out individual values (or at least some of them), for example taking the middle row of the 7×7 matrix above:

dominance

Dominance (Gerzon & Barton [1]) can be expressed by a transformation matrix:

$$\begin{aligned}
\mathbf{L} &= \begin{pmatrix} \frac{t+t^{-1}}{2} & \frac{t-t^{-1}}{2} & 0 & 0 \\ \frac{t-t^{-1}}{2} & \frac{t+t^{-1}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} && (11) \\
&= \begin{pmatrix} \csc\delta & -\cot\delta & 0 & 0 \\ -\cot\delta & \csc\delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} && (12)
\end{aligned}$$

(For convenience these are expressed in SN3D[12]. The upper form is the more usual. Their derivation and interconversion is described by Chapman [3].)

The conventional form, does though lead to infinite amplitude levels, for some values. This may result in some 'clipping' of the reproduced soundfield. Malham ([19]) proposed the upper of the following two matrices (with $-1 \geq \mu \geq 1$), which we rewrite in a geometric form (the lower of the two).

$$\begin{aligned}
\mathbf{D} &= \begin{pmatrix} 1 & \mu & 0 & 0 \\ \mu & 1 & 0 & 0 \\ 0 & 0 & \sqrt{1-\mu^2} & 0 \\ 0 & 0 & 0 & \sqrt{1-\mu^2} \end{pmatrix} && (16) \\
&= \begin{pmatrix} 1 & -\cos\delta & 0 & 0 \\ -\cos\delta & 1 & 0 & 0 \\ 0 & 0 & \sin\delta & 0 \\ 0 & 0 & 0 & \sin\delta \end{pmatrix} && (17)
\end{aligned}$$

$B'_{12} = -a.B_{13} + b.B_{15}$, or using x, y, z for u_x, etc.
$(\sqrt{7}/2).z'(2(z')^2 - 3(x')^2 - 3(y')^2) = -a.\sqrt{21/8}.x.(4z^2 - x^2 - y^2) + b.\sqrt{35/8}.x(x^2 - 3y^2)$
if we substitute z' by x, etc., and simplify:
$x^2(2\sqrt{2} - \sqrt{5}b - \sqrt{3}a) + y^2(-3\sqrt{2} + 3\sqrt{5}b - \sqrt{3}a) + (4\sqrt{3}a - 3\sqrt{2}) = 0$
For this to be true for all x, y, z:
$a = \sqrt{3/8}$ and $b = \sqrt{5/8}$.
These values agree with Zotter's published tables.

[12] A formal account of normalisations schemes, etc. is given by Daniel[8]. For our purposes, here,

$$B_n^{(SN3D)} = \alpha_l \cdot B_n^{(N3D)} \qquad (13)$$

where l is the degree of the component, and

$$\alpha_l = 1/\sqrt{2l+1} \qquad (14)$$

and has the convenience that (in SN3D) for a plane wave

$$\sum_{a=0}^{0}(B_a)^2 = \sum_{b=1}^{3}(B_b)^2 = \sum_{c=4}^{8}(B_c)^2 = \ldots = i^2 \qquad (15)$$

It will be obvious that

$$\mathbf{D} = \csc(\delta) \cdot \mathbf{L}$$

The above matrices are based on thus published for FuMa channel sequence, with the thought that these will be familiar to most readers, but are valid irrespective of channel sequence —the direction of the dominance is all that is affected. For a frontwards dominance, then with ACN–sequence, we wish to consider:

$$\begin{pmatrix} 1 & 0 & 0 & -\cos\delta \\ 0 & \sin\delta & 0 & 0 \\ 0 & 0 & \sin\delta & 0 \\ -\cos\delta & 0 & 0 & 1 \end{pmatrix} \qquad (18)$$

These results are fairly trivial, but set out here as we will return to dominance in greater detail in a later section.

decoding

Furse's description [23] of some channels as "discarded ambiguous harmonic"s (e.g. B_6 when decoding to a cube) has been the subject of discussion for some time.

Workers at Graz, with their interest in partial spheres have also reviewed which channels are 'redundant' for some situations [14, 22].

There are obvious parallels with the study here. Figure 2 is reproduced from [14] (and similar concepts are in [22]). In this figure all the channels that have z in their f_n are feint.

However, as discussion about partial signal sets is currently very active, now is perhaps not the time to explore applying the methods here to them. This should though be a fertile area for the future.

5 EXTENDING TO ARBITRARY ANGLES

The above discussion has been about utilising the symmetries of spherical harmonics in relation to the x–, y– and z–axes. If the approach of substituting for u_x, u_y and u_z is extended, but without consideration of symmetries a variety of interesting results can be obtained.

The following discussion, is however limited to transformation matrices for plane waves.

Chapman & Cotterell [4] state:

> **Proposition 1** Any valid ambisonic transformation must be valid for all possible ambisonic signal sets, and therefore must be valid for an ambisonic signal set that represents a planewave.

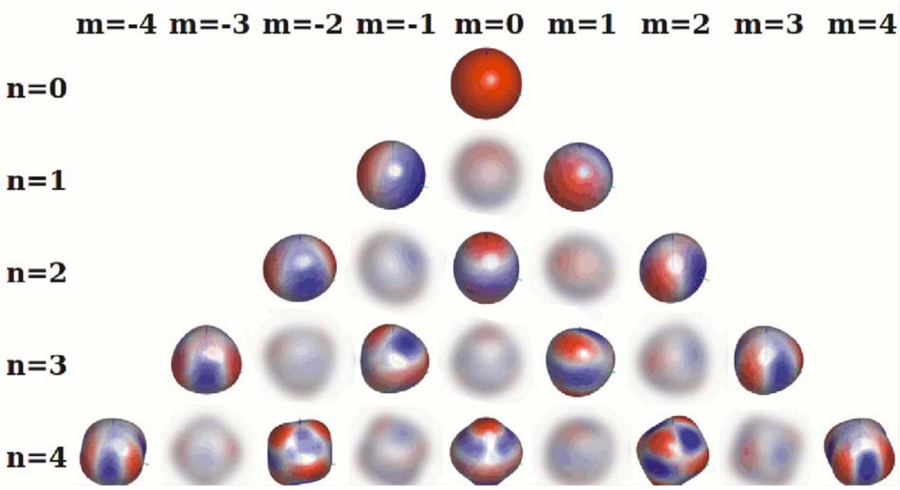

Figure 2: A consideration of SH symmetries through the horizontal plane when designing hemi-spherical loudspeaker rigs [14]. Graphic courtesy of and © Sontacchi, Zotter & Höldrich. (Note the authors use n for degree, which in the body of this paper is l.)

For block diagonal sparse matrices (which all transformation matrices, except dominance, appear to be) then a transformation may be reduced to:

$$\begin{pmatrix} u'_y \\ u'_z \\ u'_x \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \times \begin{pmatrix} u_y \\ u_z \\ u_x \end{pmatrix} \tag{19}$$

5.1. Rotation

5.1.1 A historical note on rotation matrices for higher orders

The history of spherical harmonic rotations would be a fascinating one. Here a few reflections are given as to rotations in ambisonics.

Spherical harmonics were developed in 1784 (Laplace). The work of Gegenbauer in the late Nineteenth Century (see [3]) allowed for hyperspherical harmonics of n–dimensional universes, but … as recorded below, in 2003 Malham was reporting his unsuccessful attempts at advancing SH rotation beyond second–degree. It would be interesting to know whether other SH users had no use for rotation and thus the matter was of little concern, or, if not why this area was so empty.

Even now, the previous absence of solutions to this problem is not as widely known as it ought to be (and now with solution probably never will be!). Many users have practically either not got beyond second order, or not gone beyond panthophony, or just not felt the need for non-yaw rotations.[13]

We present here a solution that is both remarkably simple (in conception) and remarkably tedious (to execute).

It is thanks to Franz Zotter that this is published here. In an aside the author had stated that this was possible but "tedious" and as Zotter was working on an automatable method[14] there seemed little point pursuing this approach. His curiosity and prompting led to this (more) formal note. With hindsight, there are places for algebraic/geometric solutions as against numeric ones —not least if it is desired to apply a progressive rotation to a soundfield.

It seems likely that the method can be extended to arbitrary higher degrees. Though, for the reasons in the previous paragraph, no attempt has been made to prove that this is the case (considering, for example, the number of simultaneous equations, variable, …, for an arbitrary degree matrix).

The position in 2003 is well summarised by Malham [24]:

> Since, starting with second order, the harmonic shapes involved in either tilt or tumble are no longer simple, generating the matrices involved is not trivial. Deriving the second order matrices is not too difficult, although it does require a significant amount of manipulation of trigonometrical equations to arrive at the results given in Furse or Daniel. However, third and higher order harmonics is "a rather intricate task", to quote a web page (…) related to the European Union Simiulgen Esprit Open Long Term Research project (…). In this project they have investigated the use

[13] As this paper was being drafted, one knowledgeable ambisonist was surprised to hear that there was a problem.

[14] see: previous section of this paper

of spherical harmonics for defining directional illumination of visual rendering systems, a clearly related task. They note that no solution to the problem of simple generation of the required rotational matrices had been found in 1995, but that this had been solved by 2000, the date of the web page. Unfortunately they give no further details, either on the web page or in the publicly available documentation from the project. However, a search of the literature in another field which uses spherical harmonics extensively, Chemical Physics, yielded a paper by Choi, Ivanic, Gordon and Ruedenberg, (Choi et al, 1999) which gives a stable recursive formula for rotations of spherical harmonics which appears to be adaptable to the conventions used in Ambisonics. Work is ongoing to apply this to software capable of working at arbitrary order.

What interest there has been in rotation, has certainly been shared (as Malham indicated) by our colleagues interested in lighting and shading for virtual images. Green (2003, [16]) in a very readable introduction to this application states that SH lighting was only introduced the previous year[15]. He discuses rotation (pages 21–26 and 46-47), not least detailing personal communications with Choi (including the latter's famous "Complex makes life easier! " comment (p.26)). Pinchon & Hoggan [25] made an interesting proposal in 2006, but the present author was not able to implement it. Instead the method presented here –if it is valid– uses naïve elementary algebra (including the avoidance of complex numbers). It is not though, without its disadvantages.

5.1.2 First degree rotation

The classic rotation matrices are:

$$\mathbf{R_{z,l=1}}(\gamma) = \begin{pmatrix} \cos(\gamma) & 0 & \sin(\gamma) \\ 0 & 1 & 0 \\ -\sin(\gamma) & 0 & \cos(\gamma) \end{pmatrix} \tag{20}$$

$$\mathbf{R_{y,l=1}}(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{pmatrix} \tag{21}$$

$$\mathbf{R_{x,l=1}}(\gamma) = \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{22}$$

That is *yaw, pitch* and *roll* respectively.

[15] attributing the advance to Sloan, Kautz & Snyder

It is a trivial rewriting, but to maintain consistency with the earlier part of this paper, these may be stated as:

yaw

$$\begin{aligned} y' &\leftarrow y \cdot \alpha + x \cdot \beta \\ z' &\leftarrow z \\ x' &\leftarrow x \cdot \alpha y \cdot \beta \end{aligned}$$

pitch

$$\begin{aligned} y' &\leftarrow y \\ z' &\leftarrow z \cdot \alpha - x \cdot \beta \\ x' &\leftarrow x \cdot \alpha + z \cdot \beta \end{aligned}$$

roll

$$\begin{aligned} y' &\leftarrow y \cdot \alpha - z \cdot \beta \\ z' &\leftarrow z \cdot \alpha + y \cdot \beta \\ x' &\leftarrow x \end{aligned}$$

where $\alpha = \cos(\gamma)$ and $\beta = \sin\gamma$.

5.2. Second degree rotation

There are well accepted matrices for arbitrary second degree yaw, pitch and roll. See for example Malham [18] or Daniel [12].

Though care must be taken with matrices from the literature to determine whether a (Condon-Shortley) phase error has been introduced (usually in 'pitch').

5.2.1 Higher degrees

For a plane wave:

$$B_n = Y_n.S \tag{23}$$

where i is the input signal.

To determine the third–degree block of the pitch matrix (which is a sparse matrix). Then for each of B'_9 to B'_{15} we could write:

$$B'_o = \sum_{n=9}^{15} a_{n,o}.B_n \tag{24}$$

$$Y'_o.i = \sum_{n=9}^{15} a_{n,o}.Y_n.S \tag{25}$$

$$Y'_o = \sum_{n=9}^{15} a_{n,o}.Y_n \tag{26}$$

Now, our transformation is a pitch, and thus:

$$\begin{aligned} y' &\leftarrow y \\ z' &\leftarrow z.\cos(\gamma) - x.\sin(\gamma) \\ x' &\leftarrow x.\cos(\gamma) + z.\sin(\gamma) \end{aligned}$$

which we may rewrite as:

$$
\begin{array}{rcl}
y' & \leftarrow & y \\
z' & \leftarrow & z.\alpha - x.\beta \\
x' & \leftarrow & x.\alpha + z.\beta
\end{array}
$$

for convenience.

Substituting in B'_9 to B'_{15} and taking $S = 1$ (this is just for convenience as if we use i it appears on both sides of a later equation, and cancels out), then we obtain the results in table 5.2.1.

If we take B'_{12} as an example, by inspection we can write

$$
\begin{array}{rcll}
Y'_{12} & = & 0.Y_9 + 0.Y_{10} + 0.Y_{11} + a.Y_{12} + b.Y_{13} & (27) \\
 & & +c.Y_{14} + d.Y_{15} & (28) \\
 & = & a.Y_{12} + b.Y_{13} + c.Y_{14} + d.Y_{15} & (29)
\end{array}
$$

The above is the equivalent of equation 26, but as we are considering only one row we have simplified the matrix element values (using $a \ldots d$). We have also said that the first three elements in this row must be zero. (B'_{12} is of the form $e.z^3 + f.x^2z + g.xz^2 + h.x^3 + i.x + j.z$ and there is no way that B_9 to Y_{11} can contribute towards that. (Assigning matrix element values to the first three elements of the row, would result in them being determined as 0 later, and a doubtful reader may expand the technique below with them and observe that happening $\ldots$.))

Generalising the above paragraph we may write the whole matrix block as:

$$
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & * & * & * & * \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix} \tag{30}
$$

Upto this point, this approach seems susceptible to automation. Solving the resulting equations (an example follows) does not, at first sight, seem to be (though the author makes no claim to expertise in computational algebra software).

Now we may rewrite equation 29 as

$$
\begin{array}{rcll}
z'(5(z')^2 - 3) & = & a.z(5z^2 - 3) + b.x\sqrt{\frac{3}{2}}(5z^2 - 1) + c.z\sqrt{15}(x^2 - y^2) & (31) \\
 & & +d.x\sqrt{\frac{5}{2}}(x^2 - 3y^2) & (32) \\
 & = & \ldots & (33) \\
 & = & x(-b.\sqrt{\frac{3}{2}} - d.3\sqrt{\frac{5}{2}}) + z(\ldots) + x^3(\ldots) + z^3(\ldots) & (34) \\
 & & +xz^2(\ldots) + x^2z(c.2\sqrt{15}) & (35)
\end{array}
$$

We now turn to the left hand side. For a pitch we may write:

$$
\begin{array}{rcl}
y' & \leftarrow & y \\
z' & \leftarrow & z.\cos(\gamma) - x.\sin(\gamma) \\
x' & \leftarrow & x.\cos(\gamma) + z.\sin(\gamma)
\end{array}
$$

$$
\begin{array}{rcl}
B'_9 & = & \sqrt{\frac{35}{8}}.y'.(3(x')^2 - (y')^2) \\
 & = & \sqrt{\frac{35}{8}}.y.(3(x.\alpha + z.\beta)^2 - (y)^2) \\
 & = & \sqrt{\frac{35}{8}}.(x^2y(3\alpha^2) + yz^2(3\beta^2) + xyz(6\alpha\beta) \\
 & & +y^3(-1)) \\
B'_{10} & = & \sqrt{105}.z'.x'.y' \\
 & = & \sqrt{105}.(z.\alpha - x.\beta).(x.\alpha + z.\beta).y \\
 & = & \sqrt{105}.(xyz(\alpha^2 - \beta^2) + x^2y(-\alpha\beta) + yz^2(\alpha\beta)) \\
B'_{11} & = & \sqrt{\frac{21}{8}}.y'.(5(z')^2 - 1) \\
 & = & \sqrt{\frac{21}{8}}.y.(5(z.\alpha - x.\beta)^2 - 1) \\
 & = & \sqrt{\frac{21}{8}}(x^2y(5\beta^2) + yz^2(5\alpha^2) + xyz(-10\alpha\beta) \\
 & & +y(-1)) \\
B'_{12} & = & \frac{\sqrt{7}}{2}.z'.(5(z')^2 - 3) \\
 & = & \frac{\sqrt{7}}{2}.(z.\alpha - x.\beta).(5(z.\alpha - x.\beta)^2 - 3) \\
 & = & \frac{\sqrt{7}}{2}(z^3(5\alpha^3) + x^2z(15\alpha\beta^2) + xz^2(-15\alpha^2\beta) \\
 & & +x^3(-5\beta^3) + x(3\beta) + z(-3\alpha)) \\
B'_{13} & = & \sqrt{\frac{21}{8}}.x'.(5(z')^2 - 1) \\
 & = & \sqrt{\frac{21}{8}}.(x.\alpha + z.\beta).(5(z.\alpha - x.\beta)^2 - 1) \\
 & = & \ldots \\
B'_{14} & = & \frac{\sqrt{105}}{2}.z'.((x')^2 - (y')^2) \\
 & = & \frac{\sqrt{105}}{2}.(z.\alpha - x.\beta).((x.\alpha + z.\beta)^2 - (y)^2) \\
 & = & \ldots \\
B'_{15} & = & \sqrt{\frac{35}{8}}.x'.((x')^2 - 3.(y')^2) \\
 & = & \sqrt{\frac{35}{8}}.(x.\alpha + z.\beta).((x.\alpha + z.\beta)^2 - 3.y^2) \\
 & = & \ldots
\end{array}
$$

Table 4: Values of B'_9 to B'_{15} for an arbitary pitch, if $\mathbf{B}$ is a plane wave. For convenience the *input signal* of the plane wave (S) is here taken to be 1 (for arbitary values of i then each of the above values needs multiplying by S (as the values derived above are used in situations where S, if used, would cancel out, this approach (slightly) simplifies the equations!)). (Expressing the spherical harmonics in 'equal powers' (see introductory text of this paper) would also make the above simpler.)

We can write $z' = \alpha.x + \beta.z$, where $\alpha = -\sin(\gamma)$ and $\beta = \cos(\gamma)$ and obviously $\alpha^2 + \beta^2 = 1$.

We may rewrite the LHS of equation 35 as follows:

$$
\begin{aligned}
(z')(5(z')^2 - 3) &= (\alpha.x + \beta.z)(5(\alpha.x + \beta.z)^2 - 3) && (36)\\
&= \ldots && (37)\\
&= x(-3\alpha) + z(\ldots) + x^3(\ldots) && (38)\\
&\quad +z^3(\ldots) + xz^2(\ldots) && (39)\\
&\quad +x^2 z(15\alpha^2\beta) && (40)
\end{aligned}
$$

As these equations must be true for any/all x, y, z (with the proviso that $x^2 + y^2 + z^2 = 1$ always) then we may write:

$$
\begin{aligned}
-b.\sqrt{\frac{3}{2}} - d.3\sqrt{\frac{5}{2}} &= -3\alpha && (41)\\
\ldots &= \ldots && (42)\\
\ldots &= \ldots && (43)\\
\ldots &= \ldots && (44)\\
\ldots &= \ldots && (45)\\
c.2\sqrt{15} &= 15\alpha^2\beta && (46)
\end{aligned}
$$

From which it follows that:

$$
\begin{aligned}
a &= \beta(2 - 5\alpha^2)/2 && (47)\\
b &= \sqrt{3/8}.\alpha.(4 - 5\alpha^2) && (48)\\
c &= (\sqrt{15}\alpha^2\beta)/2 && (49)\\
d &= \sqrt{5/8}\alpha^3 && (50)
\end{aligned}
$$

or

$$
\begin{aligned}
a &= \cos(\gamma)(2 - 5\sin^2(\gamma))/2 && (51)\\
b &= -\sqrt{3/8}.\sin(\gamma).(4 - 5\sin^2(\gamma)) && (52)\\
c &= (\sqrt{15}\sin^2(\gamma)\cos(\gamma))/2 && (53)\\
d &= -\sqrt{5/8}\sin^3(\gamma) && (54)
\end{aligned}
$$

As ever, as $\sin^2(\gamma) + \cos^2(\gamma) = 1$, these can be written in a multitude of variants.

Putting $\gamma = 270^o$, then:

$$
\begin{aligned}
a &= 0 && (55)\\
b &= -\sqrt{3/8} && (56)\\
c &= 0 && (57)\\
d &= \sqrt{5/8} && (58)
\end{aligned}
$$

Whereas Zotter has in *Ry90_03* (Zotter uses clockwise rotation) `0.0000000e+00 -6.1237244e-01 0.0000000e+00 7.9056942e-01`, which is in agreement.

(It is intended to publish a set of transformation matrices in acn order [17], after this conference.)

5.3. Dominance

For dominance (in SN3D) we have (using $\mathbf{D}$ from above (matrix 18)):

$$\mathbf{B}' = \mathbf{D} \times \mathbf{B} \tag{59}$$

and we may write, for a plane wave:

$$\mathbf{B} = \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} S \\ y \cdot S \\ z \cdot S \\ x \cdot S \end{pmatrix} \tag{60}$$

and:

$$\mathbf{B}' = \begin{pmatrix} B'_0 \\ B'_1 \\ B'_2 \\ B'_3 \end{pmatrix} = \begin{pmatrix} S' \\ y' \cdot S' \\ z' \cdot S' \\ x' \cdot S' \end{pmatrix} \tag{61}$$

Thus:

$$
\begin{aligned}
S' &= S(1 - x \cdot \cos\delta) && (62)\\
y' \cdot S' &= y \cdot \sin\delta \cdot S && (63)\\
z' \cdot S' &= z \cdot \sin\delta \cdot S && (64)\\
x' \cdot S' &= S(x - \cos\delta) && (65)
\end{aligned}
$$

Substituting

$$\frac{S}{S'} = \frac{1}{1 - x \cdot cos\delta} \tag{66}$$

$$
\begin{aligned}
y' &= y \cdot \frac{\sin\delta}{1 - x \cdot cos\delta} && (67)\\
z' &= z \cdot \frac{\sin\delta}{1 - x \cdot cos\delta} && (68)\\
x' &= \frac{x - \cos\delta}{1 - x \cdot cos\delta} && (69)
\end{aligned}
$$

Now we can, in principle, go ahead and substitute to determine higher order dominance matrices.

This approach is faced with two problems.

1. Whilst it is possible to generate a signal set with applied dominance for a plane wave, to do so one needs $(Sxyz)$. One can determine $\mathbf{B}$ from $(Sxyz)$, but *not* the inverse.[16]

[16] This involves dividing one audio signal by another, and audio signals frequently pass through zero: see this lecture's slides for n example.

2. Whilst one can produce a dominance transformed signal set ($\mathbf{B}'$) of any order from $(Sxyz)$ there is no proof that one can do so for a non planewave. This model might produce a method *if* the quest was not futile.

In reality applying this method results in transformation matrices which are not linear. Divisions by $1 - x \cdot cos\delta$, even if they were valid are in practice impossible

6 ACKNOWLEDGEMENTS

Fons Adriaensen for sparking my interest in direction cosines. Philip Cotterell for his work on them and discussion.

Franz Zotter kindly read an early draft of this manuscript and suggested many improvements, regrettably these are probably now lost amongst the additions and changes that have occurred since.

The interest and encouragement of Philip Cotterell and Franz Zotter in this work is much appreciated.

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[6] See `http://ambisonics.iem.at`

[7] *pers. comm.*

[8] Normalisation schemes are described in Daniel [9] at pages 155–157 (especially table 3.2 on page 156)..

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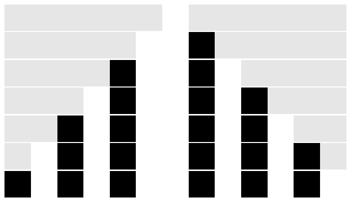
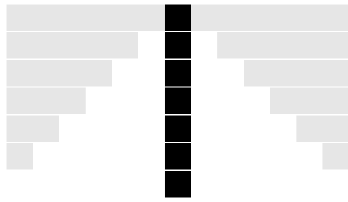
A. ATLAS

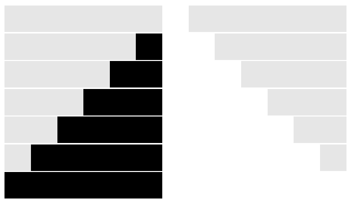


(a) **x**

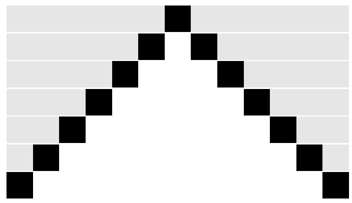
(d)



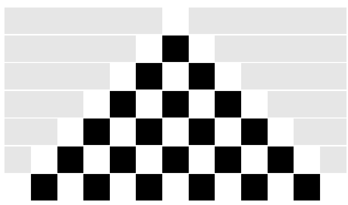
(f) zonal harmonics ($m = 0$)



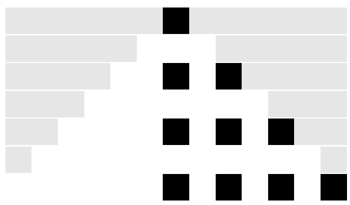
(b) **y**



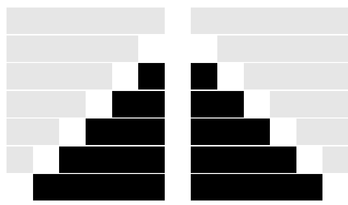
(g) sectoral harmonics ($m = \pm l$)



(c) **z**



(e) neither x nor y nor z



(h) tesseral harmonics

(d) depicts the empty grid. Columns are *degree*, that is $l = 0$ to $l = 6$. Columns are *degree*, that is $m = -6$ to $m = 6$.

(a), (b), (c) depict spherical harmonics where f_n contains x, y or z respectively. (If, for example $f_n = x \cdot y$ then the SH will be indicated in both *(a)* and *(b)*.

(e) depicts spherical harmonics where $f_n = 1$.

For comparison the clessification into zonal, sectoral and tesseral harmonics is shown in *(f), (g), (h)*.

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Figure 3: Graphical rendering of Table 3, with the addition of classification of SHs into zonal, sectoral and tesseral. Subfigure *(c)* gives the same information as Figure 2.